LECTURE NOTES ON DIFFERENTIAL EQUATIONS

By

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Abstract
In this lecture note, we give detailed explanation and set of problems on differential equations.

Topic Covered: Definition of differential equations. Order and degree of a differential equation. Formation of differential equation. Solution of first order and first degree differential equation by variable separation method (simple problems).

1. History

Differential equations have been studied by mathematicians since ancient times. The ancient Greek mathematician Archimedes used methods of integral calculus to calculate areas, volumes, and other geometric properties. He also used infinitesimals to study the motion of curves. In the 12th century, Indian mathematician Bhaskara II used differential equations to solve astronomical problems.

During the 16th century, French mathematician Francois Viete used differential equations to solve algebraic equations and developed the concept of separation of variables. In the 17th century, English mathematician Isaac Newton developed his famous laws of motion and used them to develop calculus and the concept of fluxions, which are now known as derivatives.

In the 18th century, Swiss mathematician Leonhard Euler solved ordinary differential equations and developed the concept of integrating factors, which are used to solve linear equations. In the 19th century, French mathematician Joseph-Louis Lagrange developed the theory of

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linear differential equations. He also developed the theory of partial differential equations, which allowed for the solution of equations involving multiple variables.

The 20th century saw the development of more sophisticated methods of solving differential equations, such as numerical methods and computer-aided solutions. Modern differential equations are used to solve problems in many areas, including physics, engineering, astronomy, economics, and biology. Differential equations are used to model complex systems, such as climate and weather, and to analyze the behavior of such systems. They are also used in the design of control systems and in robotics. Differential equations are also essential in the study of chaos theory, which studies the behavior of complex systems.

2. Introduction

Differential equations are mathematical equations that involve the derivatives of a function. They are used to describe the behavior of physical systems and phenomena, such as the motion of objects, the flow of heat, or the diffusion of a substance. Differential equations are essential tools in the fields of engineering, physics, economics, and other sciences.

Differential equations can be divided into two main classes: ordinary differential equations (ODEs) and partial differential equations (PDEs). ODEs involve functions of one variable (usually time) and their derivatives. PDEs involve functions of multiple variables and their partial derivatives.

For diploma students, it is important to understand the basics of differential equations. This includes the basics of ODEs and PDEs, their various types, and their applications. It is also important to understand the methods used to solve differential equations, such as separation of variables and numerical methods.

In addition, diploma students should understand the significance of differential equations and their applications in various fields. This includes understanding how differential equations can be used to model physical systems and predict their behavior. It also includes understanding how differential equations can be used to solve problems in engineering, economics, and other areas.

Finally, diploma students should also be familiar with the different types of software and packages used to solve differential equations. This includes understanding how to use software such as MATLAB or Maple to solve differential equations. This will help diploma students to be better prepared to use differential equations in their future career.

3. Differential Equation

Definition 1. An equation involving derivatives of the dependent variable with respect to independent variable(s) is known as a differential equation.

Some examples of differential equations are,

- $\dfrac{dy}{dx} = 4x^3$
- $\dfrac{dy}{dx} + \dfrac{4}{x}y = x^2$
- $\dfrac{dy}{dx} = \dfrac{x+2y}{x-y}$
- $\dfrac{d^4y}{dx^4} = 81y$
- $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 65\cos 2x$
- $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{c} = E\sin \omega t$
- $2\dfrac{d^2y}{dx^2} + 5x\left(\dfrac{dy}{dx}\right)^3 - 6y = \log x$

3.1. Types of Differential Equations

There are two types of differential equations,

- Ordinary differential equations.

- Partial differential equations.

3.2. Ordinary Differential Equations

A differential equation involving only one independent variable is called an ordinary differential equation. For example,

- $\dfrac{dy}{dx} = 4x^3$
- $\dfrac{dy}{dx} = \dfrac{x+2y}{x-y}$
- $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 65\cos 2x$
- $2\dfrac{d^2y}{dx^2} + 5x\left(\dfrac{dy}{dx}\right)^3 - 6y = \log x$

3.3. Partial Differential Equations

A differential equation involving two or more independent variables and having partial differential coefficient is called a partial differential equation. For example,

- $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$
- $\dfrac{\partial^2 z}{\partial x^2} + k^2\dfrac{\partial^2 z}{\partial y^2} = 0$
- $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

4. Order and Degree of Differential Equation

4.1. Order of Differential Equation

The order of a differential equation is simply the order of the highest order derivative explicitly appearing in the equation.

The equations

$$\frac{dy}{dx} = 4x^3, \qquad \frac{dy}{dx} + \frac{4}{x}y = x^2 \quad \text{and} \quad \frac{dy}{dx} = \frac{x+2y}{x-y}$$

are all first-order differential equations. So is

$$\frac{dy}{dx} + 3y^2 = y\left(\frac{dy}{dx}\right)^4,$$

despite the appearance of the higher powers — $\frac{dy}{dx}$ is still the highest order derivative in this equation, even if it is multiplied by itself a few times.

The equations

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 65\cos 2x \qquad \text{and} \qquad 4x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + [4x^2 - 1]y = 0$$

are second-order equations, while

$$\frac{d^3y}{dx^3} = e^{4x} \qquad \text{and} \qquad \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = x^2$$

are third-order equations.

4.2. Degree of Differential Equation

The degree of a differential equation is the highest power (positive integer only) of the highest order derivative after removing the radicals and fractions from the equation.

Remark: Degree of a differential equation is defined if it is a polynomial equation in its derivatives.

Example 1. Find the degree of the differential equation

$$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3. \qquad [2017]$$

Solution 1. In the given problem, the highest order derivative is $\frac{d^2y}{dx^2}$, and the highest power of $\frac{d^2y}{dx^2}$ is 2. Therefore, the degree of the given differential equation is 2.

Example 2. Find the degree of the differential equation

$$(x^2+1)\frac{dy}{dx}+2xy=4x^2. \qquad [2013]$$

Solution 2. The first order derivative $\frac{dy}{dx}$ is the highest derivative in the given equation and the highest degree of $\frac{dy}{dx}$ is 1. Hence the degree of the given differential equation is 1.

Example 3. Find the degree of the differential equation

$$\frac{dy}{dx}+\sin\frac{dy}{dx}=0. \qquad [2018]$$

Solution 3. The given differential equation involves the highest derivative of first order. So, it is of order 1. It is not a polynomial equation in $\frac{dy}{dx}$. So, its degree cannot be defined.

Example 4. Find the order and degree of the differential equation

$$y=x\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad [2020\ (ODD)]$$

Solution 4. The given differential equation is,

$$y=x\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

or,

$$y-x\frac{dy}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Squaring on both sides, we get

$$\begin{aligned}
& \left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \\
\Longrightarrow \quad & y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \\
\Longrightarrow \quad & x^2\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - 1 = 0 \\
\Longrightarrow \quad & (1 - x^2)\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + (1 - y^2) = 0.
\end{aligned}$$

This is a differential equation of order 1 because the highest derivative is $\frac{dy}{dx}$. The highest degree of $\frac{dy}{dx}$ is 2. Hence the order of the given differential equation is 1 and the degree is 2.

Example 5. Find the order and degree of the differential equation

$$1 + \left(\frac{dy}{dx}\right)^2 = \left[2 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}.$$

Solution 5. The given differential equation is,

$$1 + \left(\frac{dy}{dx}\right)^2 = \left[2 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

Squaring on both sides, we get

$$\begin{aligned}
& \left[1 + \left(\frac{dy}{dx}\right)^2\right]^2 = \left[2 + \left(\frac{dy}{dx}\right)^2\right]^3 \\
\Longrightarrow \quad & 1 + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^4 = \left[2 + \left(\frac{dy}{dx}\right)^2\right]^3.
\end{aligned}$$

The highest derivative in the above equation is $\frac{dy}{dx}$. Hence the order is 1 and degree is 6.

4.3. Try Exercise

Find the degree and order of each of the following differential equations.

1. $\dfrac{dy}{dx} = \sin x - y\cos x.$

2. $\left(\dfrac{dy}{dx}\right)^{\frac{3}{2}} = x.$

3. $\left(\dfrac{d^2y}{dx^2}\right)^2 + \cos\dfrac{dy}{dx} = 0.$

4. $\dfrac{d^4y}{dx^4} - \sin\left(\dfrac{d^3y}{dx^3}\right)^2 = 0.$

5. $\dfrac{d^{42}y}{dx^{42}} = \left(\dfrac{d^3y}{dx^3}\right)^2.$

6. $\dfrac{d^5y}{dx^5} - \cos x\dfrac{d^3y}{dx^3} = y^6.$

7. $y''' - 3\left(y''\right)^2 + \left(y'\right)^3 + \log y = 5x.$

8. $\left(y'''\right)^2 + \left(y''\right)^3 + \left(y'\right)^4 + y^5 = 0.$

9. $\left(\dfrac{dy}{dx}\right)^2 + x^3\dfrac{dy}{dx} - x^2y = 0.$

10. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y^4 = 0.$

11. $\dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = r$, where r is constant. [2013]

12. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 7y = \sin x.$ [2014]

13. $\left(\dfrac{dy}{dx}\right)^2 + \dfrac{1}{\left(\frac{dy}{dx}\right)} = 1.$ [2015]

14. $\left(\frac{d^2y}{dx^2}\right)^4 + 2y\left(\frac{d^2y}{dx^2}\right) = 0.$ [2018 (EVEN)]

15. $x\left(\frac{d^2y}{dx^2}\right)^2 + y\left(\frac{dy}{dx}\right)^3 + y^2 = 0.$ [2018 (ODD)]

16. $x^2\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = \log x.$ [2019 (ODD)]

5. Formation of Differential Equation

To form a differential equation from a given function we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

Example 6. Form the differential equation represinting the family of curves $xy = a^2$, where a is an arbitrary constant.

Solution 6. Given family of curves is

$$xy = a^2$$

Differentiating both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0$$

which is the required differential equation. □

Example 7. Form the differential equation represinting the family of curves $x^2 - y^2 = a^2$, where a is an arbitrary constant.

Solution 7. Given family of curves is

$$x^2 - y^2 = a^2$$

Differentiating both sides w.r.t. x, we get

$$2x - 2y\frac{dy}{dx} = 0$$

$$\implies \quad x - y\frac{dy}{dx} = 0.$$

which is the required differential equation. □

Example 8. Form the differential equation represinting the family of curves $y = mx$, where m is an arbitrary constant.

Solution 8. Given family of curves is

$$y = mx$$

or,

$$\frac{y}{x} = m$$

Differentiating both sides w.r.t. x, we get

$$\frac{x\frac{dy}{dx} - y}{x^2} = 0$$

$$\implies \qquad x\frac{dy}{dx} - y = 0.$$

which is the required differential equation. □

Example 9. Form the differential equation by eliminating the arbitrary constant from $y = \frac{a+x}{x^2+1}$.

Solution 9. Given family of curves is

$$y = \frac{a+x}{x^2+1}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (a+x)(2x)}{(x^2+1)^2},$$

$$= \frac{(x^2+1) - (x^2+1)y\ 2x}{(x^2+1)^2},$$

[From the given equation, $(x^2+1)y = a + x$]

$$= \frac{(x^2+1)(1-2xy)}{(x^2+1)^2},$$

$$= \frac{1-2xy}{x^2+1}$$

$$\implies \qquad \frac{dy}{dx}(x^2+1) + 2xy - 1 = 0.$$

which is the required differential equation. □

Example 10. Form the differential equation represinting the family of curves $y^2 = 4a(x+a)$, where a is an arbitrary constant.

Solution 10. Given family of curves is

$$y^2 = 4a(x+a)$$
$$\implies \quad y^2 = 4ax + 4a^2 \tag{1}$$

Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$
$$\implies \quad \frac{y}{2}\frac{dy}{dx} = a$$

Substituting the value of a in Equation (1), we get

$$y^2 = 4\left(\frac{y}{2}\frac{dy}{dx}\right)x + 4\left(\frac{y}{2}\frac{dy}{dx}\right)^2,$$
$$\implies \quad y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2,$$

which is the required differential equation. □

Example 11. Form the differential equation represinting the family of curves $(x-a)^2 - y^2 = 1$, where a is an arbitrary constant.

Solution 11. Given family of curves is

$$(x-a)^2 - y^2 = 1, \tag{1}$$
$$\implies \quad x^2 + a^2 - 2ax - y^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$2x - 2a - 2y\frac{dy}{dx} = 0,$$
$$\implies \quad 2a = 2x - 2y\frac{dy}{dx},$$
$$\implies \quad a = x - y\frac{dy}{dx}.$$

Substituting the value of a in Equation (1), we get

$$\left(x - x + y\frac{dy}{dx}\right)^2 - y^2 = 1,$$

$$\implies \quad y^2\left(\frac{dy}{dx}\right)^2 = 1,$$

which is the required differential equation. □

Example 12. Form the differential equation represinting the family of curves $y = \tan^{-1}x + ae^{-\tan^{-1}x}$, where a is an arbitrary constant.

Solution 12. Given family of curves is

$$y = \tan^{-1}x + ae^{-\tan^{-1}x}$$

$$\implies \quad y - \tan^{-1}x = \frac{a}{e^{\tan^{-1}x}}$$

$$\implies \quad e^{\tan^{-1}x}\left(y - \tan^{-1}x\right) = a$$

Differentiating both sides w.r.t. x, we get

$$e^{\tan^{-1}x}\left(\frac{dy}{dx} - \frac{1}{1+x^2}\right) + \left(y - \tan^{-1}x\right)\left(\frac{e^{\tan^{-1}x}}{1+x^2}\right) = 0,$$

$$\implies \quad e^{\tan^{-1}x}\left[\frac{dy}{dx} - \frac{1}{1+x^2} + \frac{y - \tan^{-1}x}{1+x^2}\right] = 0,$$

$$\implies \quad \frac{dy}{dx} + \frac{(-1 + y - \tan^{-1}x)}{1+x^2} = 0,$$

$$\implies \quad \left(1+x^2\right)\frac{dy}{dx} - 1 + y - \tan^{-1}x = 0,$$

$$\implies \quad \left(1+x^2\right)\frac{dy}{dx} + y = 1 + \tan^{-1}x,$$

which is the required differential equation. □

Example 13. Find the differential equation of the family of curves $y = ae^{3x} + be^{5x}$ for different values of a and b.

Solution 13. Given family of curves is

$$y = ae^{3x} + be^{5x} \tag{1}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3ae^{3x} + 5be^{5x} \tag{2}$$

Again differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 9ae^{3x} + 25be^{5x} \tag{3}$$

Eliminating a and b from above three equations, we get

$$\begin{vmatrix} e^{3x} & e^{5x} & -y \\ 3e^{3x} & 5e^{5x} & -y' \\ 9e^{3x} & 25e^{5x} & -y'' \end{vmatrix} = 0.$$

Taking e^{3x}, e^{5x} and -1 common from first, second and third columns respectively, we shall have

$$-e^{3x}\, e^{5x} \begin{vmatrix} 1 & 1 & y \\ 3 & 5 & y' \\ 9 & 25 & y'' \end{vmatrix} = 0,$$

$$\Longrightarrow \qquad \begin{vmatrix} 1 & 1 & y \\ 3 & 5 & y' \\ 9 & 25 & y'' \end{vmatrix} = 0.$$

Simplifying with respect to third column, we get

$$30y - 16y' + 2y'' = 0,$$

or, $$y'' - 8y' + 15y = 0.$$

or, $$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0.$$

Example 14. Find the differential equation representing the family of curves $ax^2 + by^2 = 1$, where a and b are parameters.

Solution 14. Given family of curves is

$$ax^2 + by^2 = 1 \tag{4}$$

Differentiating above equation w.r.t. x, we get

$$ax + by\frac{dy}{dx} = 0 \tag{5}$$

Again differentiating above equation w.r.t. x, we get

$$a + b\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] = 0 \tag{6}$$

Eliminating the parameter a and b from above three equations, we get

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & y\frac{dy}{dx} & 0 \\ 1 & \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} & 0 \end{vmatrix} = 0$$

Expanding with respect to third column and simplifying, we get

$$x\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] - y\frac{dy}{dx} = 0,$$

which is the required differential equation.

5.1. Try Exercise

Form the differential equation of the following family of curves :

1. $y = mx + c$, where c is arbitrary constant. [2018 (ODD)]
2. $y = e^{mx}$, where m is arbitrary constant.

3. $y = ke^{\tan^{-1} x}$, where k is arbitrary constant.

4. $x^2 + y^2 = ax^3$, where a is arbitrary constant.

5. $x^2 + (y - b)^2 = 1$, where b is arbitrary constant.

6. $y = ax + 2a^2 + a^3$, where a is arbitrary constant.

7. $(x - a)^2 + 2y^2 = a^2$, where a is arbitrary constant.

8. $y = ae^{2x} + be^{3x}$, where a and b are arbitrary constants.

9. $y = a \cos 3x + b \sin 3x$, where a and b are arbitrary constants.

10. $y = a \cos(\log x) + b \sin(\log x)$, where a and b are arbitrary constants.

6. Solution of Differential Equation

Any function that satisfies a given differential equation is called a solution to that differential equation.

Example 15. $y = e^{-3x}$ is the solution of differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$$

Example 16. $y = a \cos x + b \sin x$ is the solution of differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

In above definition, "Satisfies the differential equation", means that, if you plug the function into the differential equation and compute the derivatives, then the result is an equation that is true no matter what real value we replace the variable with. And if that resulting equation is not true for some real values of the variable, then that function is not a solution to that differential equation.

Example 17. Consider the differential equation

$$\frac{dy}{dx} - 3y = 0.$$

If, we let $y(x) = e^{3x}$ (i.e., if we replace y with e^{3x}), we get

$$\begin{aligned} & \frac{d}{dx}\left[e^{3x}\right] - 3e^{3x} = 0 \\ \Longrightarrow \quad & 3e^{3x} - 3e^{3x} = 0 \\ \Longrightarrow \quad & 0 = 0, \end{aligned}$$

which certainly is true for every real value of x . So $y(x) = e^{3x}$ is a solution to our differential equation.

On the other hand, if we let $y(x) = x^3$, we get

$$\begin{aligned} & \frac{d}{dx}\left[x^3\right] - x^3 = 0 \\ \Longrightarrow \quad & 3x^2 - 3x^3 = 0 \\ \Longrightarrow \quad & 3x^2(1-x) = 0, \end{aligned}$$

which is true only if $x = 0$ or $x = 1$. But our interest is not in finding values of x that make the equation true; our interest is in finding functions of x (i.e., $y(x)$) that make the equation true for all values of x . So $y(x) = x^3$ is not a solution to our differential equation. (And it makes no sense, whatsoever, to refer to either $x = 0$ or $x = 1$ as solutions, here.)

6.1. General Solution of Differential Equation

The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution.

Example 18. $y = e^{mx}$ is the general solution of first order differential equation

$$x\frac{dy}{dx} = y\log y,$$

as it contains one orbitrary independent constant.

Example 19. $y = a\cos x + b\sin x$ is the general solution of second order differential equation

$$\frac{d^2y}{dx^2} + y = 0,$$

as it contains two orbitrary independent constants.

Note: General solution or complete solution of an n^{th} order differential equation will have n arbitrary constants.

6.2. Particular Solution of Differential Equation

The solution free from arbitrary constants is called particular solution. It is obtained from the general solution by giving particular values to the independent arbitrary constants.

Example 20. If we put $m = 3$, then $y = e^{3x}$ is the particular solution of differential equation

$$x\frac{dy}{dx} = y\log y.$$

Example 21. If we put $a = 2$ and $b = 3$, then $y = 2\cos x + 3\sin x$ is the particular solution of differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Note: A particular solution of a differential equation is not unique. Typically, a differential equation will have many different solutions. Any formula (or, set of formulas) that describes all possible solutions is called a general solution to the differential equation.

6.3. Independence of Arbitrary Constants

The arbitrary constants occuring in the general solution of a differential equation must be independent and in order to test it whether they are really independent or not we have to show that they can not be further reduced to fewer number of equivalent constants.

Example 22. Show that the function $y = ax + 2a^2$ is a solution of differential equation

$$2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$$

Solution 15. Given function is

$$y = ax + 2a^2 \tag{7}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = a$$

Substituting the value of a in Equation (7), we get

$$y = x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2$$

$$\implies \qquad 2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0,$$

which is the given differential equation. Hence, the given function is a solution of the given differential equation.

Example 23. Verify that the function $y = e^{-3x}$ is a solution of differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$$

Solution 16. Given function is

$$y = e^{-3x} \tag{1}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = -3e^{-3x} \tag{2}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

$$\Longrightarrow \quad \frac{d^2y}{dx^2} - 9e^{-3x} = 0,$$

$$\Longrightarrow \quad \frac{d^2y}{dx^2} - 3e^{-3x} - 6e^{-3x} = 0,$$

$$\Longrightarrow \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0, \qquad \text{[Using (1) and (2)]}$$

which is the given differential equation. Hence, the given function is a solution of the given differential equation.

6.4. Try Exercise

1. Verify that the function $y - \cos y = x$ is a solution of differential equation
$$(x + \cos y + y \sin y)\frac{dy}{dx} = y.$$
2. Verify that the function $y = \sqrt{1 + x^2}$ is a solution of differential equation
$$\frac{dy}{dx} = \frac{xy}{1 + x^2}.$$
3. Verify that the function $y = ax + \frac{b}{x}$ is a solution of differential equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$
4. Verify that the function $y + x + 1 = 0$ is a solution of differential equation
$$(y - x)dy - (y^2 - x^2)dx = 0.$$
5. Verify that the function $y = a\cos(\log x) + b\sin(\log x)$ is a solution of differential equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

7. First Order and First Degree Differential Equation

A differential equation of the form $Mdx + Ndy = 0$, where M and N are functions of x and y or constants is called a differential equation of first order and first degree.

Note: The general solution of such differential equation will contain only one arbitrary constant.

7.1. Variables Separable Method

If in an equation it is possible to separate variables completely, i.e., terms containing y should remain with dy and terms containing x should remain with dx, then the variables are said to be separable.

Thus the general form of such equation is

$$f(x)dx = g(y)dy$$

Integrating both sides, we get

$$\int f(x)dx = \int g(y)dy + c$$

is the solution. The constants of integration that appears on both the sides are combined together to give just one arbitrary constant c.

Example 24. Solve the differential equation

$$\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}.$$

Solution 17. Given equation is,

$$\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$$

$$\implies \quad \frac{dy}{dx} = e^{-2y}\left(e^{3x} + x^2\right)$$

Separating the variables, we have

$$e^{2y}dy = \left(e^{3x} + x^2\right)dx$$

Integrating both sides, we get

$$\int e^{2y} dy = \int \left(e^{3x} + x^2\right) dx$$

$$\implies \quad \frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c$$

$$\implies \quad 3e^{2y} = 2\left(e^{3x} + x^3\right) + c$$

which is the required solution.

Example 25. Solve the differential equation

$$(x+y)^2 \frac{dy}{dx} = a^2.$$

Solution 18. Given differential equation is,

$$(x+y)^2 \frac{dy}{dx} = a^2.$$

Rearranging the terms, we have

$$\left(\frac{x+y}{a}\right)^2 = \frac{dx}{dy}.$$

Let

$$\frac{x+y}{a} = t$$

Differentiating both sides of above w.r.t. y, we get

$$\frac{1}{a}\left[1 + \frac{dx}{dy}\right] = \frac{dt}{dy},$$

$$\implies \quad a\left(\frac{dt}{dy} - 1\right) = t^2,$$

$$\implies \quad a\frac{dt}{dy} = 1 + t^2,$$

$$\implies \quad \frac{dt}{1+t^2} = \frac{dy}{a}.$$

On integration, we get

$$\tan^{-1} t = \frac{y}{a} + c^{'},$$

$$\implies \qquad \tan^{-1}\left(\frac{x+y}{a}\right) = \frac{y}{a} + c^{'},$$

$$\implies \qquad a\tan^{-1}\left(\frac{x+y}{a}\right) = y + c,$$

which is the required solution.

Example 26. Solve the differential equation

$$x(y^2+1)dx + y(x^2+1)dy = 0.$$

Solution 19. Given differential equation is,

$$x(y^2+1)dx + y(x^2+1)dy = 0.$$

Dividing both sides by $(x^2+1)(y^2+1)$, we have

$$\frac{x}{x^2+1}dx + \frac{y}{y^2+1}dy = 0$$

Integrating both sides of above, we get

$$\frac{1}{2}\log(x^2+1) + \frac{1}{2}\log(y^2+1) = C.$$

Put $C = \dfrac{1}{2}\log A$ in above solution for further simplification, we get

$$(x^2+1)(y^2+1) = A,$$

which is the required solution.

Example 27. Find the general solution of the differential equation

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0.$$

Solution 20. Given differential equation is,

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

$$\implies \qquad \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

$$\implies \qquad \frac{1}{\sqrt{1-y^2}}dy = -\frac{1}{\sqrt{1-x^2}}dx$$

Integrating both sides of above, we get

$$\sin^{-1} y = -\sin^{-1} x + c,$$

$$\text{or,} \qquad \sin^{-1} x + \sin^{-1} y = c,$$

which is the required solution.

Example 28. Find the general solution of the differential equation

$$x\sqrt{1-y^2}\ dx + y\sqrt{1-y^2}\ dy = 0.$$

Solution 21. Given differential equation is,

$$x\sqrt{1-y^2}\ dx + y\sqrt{1-y^2}\ dy = 0$$

$$\text{or,} \qquad x\sqrt{1-y^2}\ dx = -y\sqrt{1-y^2}\ dy$$

$$\text{or,} \qquad \frac{x}{\sqrt{1-x^2}}dx = -\frac{y}{\sqrt{1-y^2}}dy$$

Integrating both sides, we get

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\int \frac{y}{\sqrt{1-y^2}}dy,$$

$$\text{or,} \qquad -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx = \frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}}dy,$$

$$\text{or,} \qquad -\sqrt{1-x^2} = \sqrt{1-y^2} + k,$$

$$\text{or,} \qquad \sqrt{1-x^2} + \sqrt{1-y^2} = -k = c \text{ (say)},$$

$$\text{or,} \qquad \sqrt{1-x^2} + \sqrt{1-y^2} = c.$$

which is the required solution.

7.2. Equations Reducible to Variable Separable

Equation of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

can be reduced to the form in which the variables are separable.

Substituting $ax + by + c = z$, we have

$$a + b\frac{dy}{dx} = \frac{dz}{dx},$$

$$\text{or,} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right).$$

Hence, the given equation becomes

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z),$$

$$\text{or,} \quad \frac{dz}{dx} - a = bf(z),$$

$$\text{or,} \quad \frac{dz}{dx} = a + bf(z).$$

Separating the variables, we get

$$\frac{dz}{a + bf(z)} = dx,$$

which can now be integrated.

Example 29. Solve the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$.

Solution 22. Given differential equation is

$$\frac{dy}{dx} + 1 = e^{x+y}$$

Putting $x + y = v$, we have

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

Substituting these values in the given differential equation, we get

$$\frac{dv}{dx} = e^v,$$

$$\text{or,} \quad e^{-v} dv = dx.$$

On integration, we get

$$-e^{-v} = x + c,$$

$$\text{or,} \quad -1 = (x + c)e^v$$

$$\text{or,} \quad (x + c)e^{x+y} + 1 = 0,$$

where c is arbitrary constant.

Example 30. Solve the differential equation $\dfrac{dy}{dx} = (2x + 3y - 4)^2$.

Solution 23. Given differential equation is

$$\frac{dy}{dx} = (2x + 3y - 4)^2$$

Putting $2x + 3y - 4 = v$, we have

$$2 + 3\frac{dy}{dx} = \frac{dv}{dx},$$

$$\text{or,} \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right).$$

Substituting these values in the given differential equation, we get

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = v^2$$

$$\text{or,} \quad \frac{dv}{dx} = 3v^2 + 2$$

$$\text{or,} \quad dx = \frac{dv}{3v^2 + 2}$$

$$= \frac{1}{3} \cdot \frac{dv}{v^2 + \frac{2}{3}}$$

Integrating both sides, we get

$$\begin{aligned} x &= \frac{1}{3}\int \frac{dv}{v^2+\left(\sqrt{\frac{2}{3}}\right)^2} + c, \\ &= \frac{1}{3}\cdot\sqrt{\frac{3}{2}}\tan^{-1}\left(\sqrt{\frac{3}{2}}v\right) + c, \\ x &= \frac{1}{\sqrt{6}}\tan^{-1}\left(\sqrt{\frac{3}{2}}\,(2x+3y-4)\right) + c. \end{aligned}$$

where c is arbitrary constant.

Example 31. Solve the differential equation $(x+y)^2\dfrac{dy}{dx} = a^2$.

Solution 24. Given differential equation is

$$(x+y)^2\frac{dy}{dx} = a^2$$

Putting $x+y=v$, we have

$$1+\frac{dy}{dx} = \frac{dv}{dx},$$

$$\text{or,} \qquad \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

Substituting these values in the given differential equation, we get

$$v^2\left(\frac{dv}{dx}-1\right) = a^2$$

$$\implies \qquad \frac{dv}{dx} - 1 = \frac{a^2}{v^2}$$

$$\implies \qquad \frac{dv}{dx} = 1 + \frac{a^2}{v^2}$$

$$\implies \qquad \frac{dv}{dx} = \frac{a^2+v^2}{v^2}$$

$$\implies \qquad dx = \frac{v^2}{a^2+v^2}\; dv$$

$$\implies \qquad dx = \left(1 - \frac{a^2}{a^2+v^2}\right) dv$$

Integrating both sides, we get

$$\begin{aligned}
&& x &= v - a^2 \cdot \frac{1}{a}\tan^{-1}\frac{v}{a} + c, \\
\implies && x &= x + y - a\tan^{-1}\frac{x+y}{a} + c, \qquad (\because v = x+y) \\
\implies && a\tan^{-1}\frac{x+y}{a} &= y + c, \\
\implies && \tan^{-1}\frac{x+y}{a} &= \frac{y+c}{a}, \\
\implies && x + y &= a\tan\left(\frac{y+c}{a}\right),
\end{aligned}$$

where c is arbitrary constant.

Example 32. Solve the differential equation $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$.

Solution 25. Given differential equation is

$$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$$

Putting $x + y = v$, we have

$$\begin{aligned}
&& 1 + \frac{dy}{dx} &= \frac{dv}{dx}, \\
\text{or,} && \frac{dy}{dx} &= \frac{dv}{dx} - 1.
\end{aligned}$$

Substituting these values in the given differential equation, we get

$$\begin{aligned}
&& \frac{dv}{dx} - 1 &= \sin v + \cos v, \\
\implies && \frac{dv}{dx} &= 1 + \sin v + \cos v, \\
\implies && dx &= \frac{dv}{\sin v + (1 + \cos v)}, \\
\implies && dx &= \frac{dv}{2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2}}.
\end{aligned}$$

Integrating both sides, we get

$$x = \frac{1}{2}\int \frac{dv}{\sin\frac{v}{2}\cos\frac{v}{2} + \cos^2\frac{v}{2}} + c.$$

Dividing numerator and denominator by $\cos^2\frac{v}{2}$, we have

$$x = \int \frac{\frac{1}{2}\sec^2\frac{v}{2}}{\tan\frac{v}{2} + 1} + c,$$

$$\text{or,} \qquad x = \log\left|1 + \tan\frac{v}{2}\right| + c,$$

$$\text{or,} \qquad x = \log\left|1 + \tan\left(\frac{x+y}{2}\right)\right| + c.$$

where c is arbitrary constant.

7.2.1. Variable Separable through Polar Tranformation

Sometimes transformation to the polar coordinates is more suitable for separation of variables. In this connection, it is convenient to remember the following differentials.

If $x = r\cos\theta$ and $y = r\sin\theta$, then

- $x\ dx + y\ dy = r\ dr$
- $x\ dy - y\ dx = r^2\ d\theta$
- $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$.

Example 33. Solve the following differential equation

$$x\ dx + y\ dy + (x^2 + y^2)dy = 0.$$

Solution 26. Given differential equation is

$$x\ dx + y\ dy + (x^2 + y^2)dy = 0.$$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\begin{aligned} r^2 &= x^2 + y^2; \\ \tan\theta &= \frac{y}{x}, \\ x\ dx + y\ dy &= r\ dr, \\ \text{and} \qquad x\ dy - y\ dx &= r^2\ d\theta. \end{aligned}$$

Therefore, the given equation becomes

$$\begin{aligned} r\ dr + r^2 dy &= 0, \\ \text{or,} \qquad \frac{dr}{r} &= -dy \end{aligned}$$

Integrating both sides, we get

$$\begin{aligned} \log r &= -y + k', \\ \text{or,} \qquad r &= e^{-y} \cdot e^{k'}, \\ \text{or,} \qquad \sqrt{x^2 + y^2} &= ke^{-y}, \\ \text{or,} \qquad x^2 + y^2 &= k^2 e^{-2y}, \\ \text{or,} \qquad x^2 + y^2 &= ce^{-2y}, \end{aligned}$$

where c is arbitrary constant.

7.3. Try Exercise

Solve the following differential equations:

1. $\dfrac{dy}{dx} = (e^x + 1)\, y.$

2. $\dfrac{dy}{dx} = e^{y+x} + e^y x^2.$

3. $\dfrac{dy}{dx} = \dfrac{x}{x^2 + 1}.$

4. $\dfrac{dy}{dx} = 1 + x + y + xy.$

5. $\tan y \; dx + \sec^2 y \tan x \; dy = 0.$

6. $\sec^2 x \tan y \; dx + \sec^2 y \tan x \; dy = 0.$

7. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right).$

8. $\dfrac{dy}{dx} = (4x + y + 1)^2.$

9. $\dfrac{dy}{dx} = \dfrac{2}{x + 2y - 3}.$

10. $\dfrac{dy}{dx} = \dfrac{(x - y) + 3}{2(x - y) + 5}.$

11. $(x - y)^2 \dfrac{dy}{dx} = a^2.$

12. $(x^2 - yx^2)\, dy + (y^2 + x^2y^2)\, dx = 0.$

13. $y \; dx - x \; dy = xy \; dx.$

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